The reconstruction of speed of sound by boundary control method on semiplane: implementation and results

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The detailed paper: http://arxiv.org/abs/1505.06176

The more shots, the more information...

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The first affirmative results in 2D acoustics with spectral inverse data were obtained by V. B. Filippov (1994) and also S. A. Ivanov and V. Yu. Gotlib (1997). A dynamical variant of BCM was implemented and tested by V.Yu.Gotlib (1999).

Acoustic equation on semiplane

We consider acoustic equation in $\Omega \times (0, T)$ where $\Omega \in R^n$, $\Gamma := \partial\Omega$,

$$\left[\frac{\partial^2}{\partial t^2} - c^2(x)\Delta\right] u(x,t) = 0,$$

$$u(x,t)|_{t=0} = 0, \quad \frac{\partial}{\partial t} u(x,t)|_{t=0} = 0, \quad u(x,t)|_{\Gamma\times[0,T]} = f(x,t).$$

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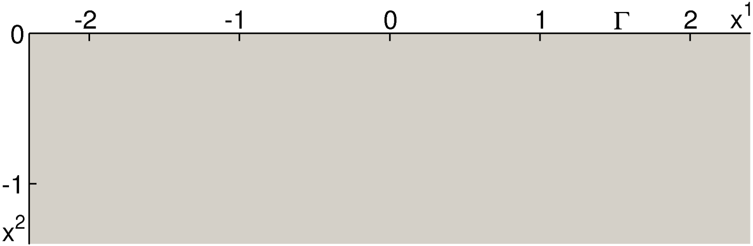

Figure: Semiplane.

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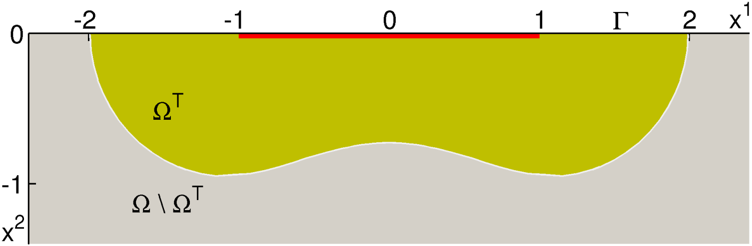

Figure: Region filled by waves.

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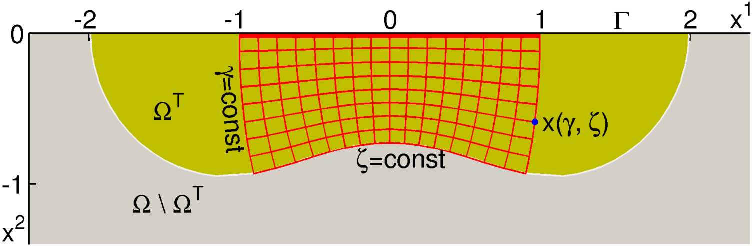

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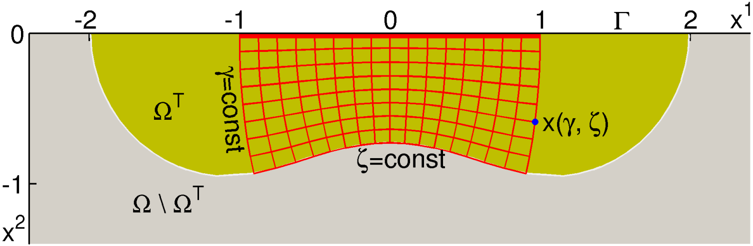

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How to recover $c(x)$ in Ω^T from measurements of the system's reaction $\frac{\partial}{\partial\nu} u(x,t)|_{\Gamma\times[0,2T]}$ on boundary controls $f(x,t)$?

Dual system

The dynamic system defined in a cylinder $\Omega \times (0, T)$, $\Sigma^T := \Gamma \times [0, T]$

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$$(f,g)_{\mathcal{F}^T} := \int_0^T \mathrm{d}t \int_\Gamma \mathrm{d}\Gamma\, f(\gamma,t)\, g(\gamma,t).$$

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For any boundary control $h \in \mathcal{F}^T$ and initial value $y \in \mathcal{H}$ the following equality holds:

$$(y, u^h(\cdot, T))_{\mathcal{H}} = (\partial v^y / \partial \nu, h)_{\mathcal{F}^T}.$$

Key elements of BCM: scalar products

Let us introduce the *connecting operator* $C^T : \mathcal{F}^T \to \mathcal{F}^T$,

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$$(C^T f)(\gamma, t) = \frac{1}{2}\left[\frac{\partial u^F}{\partial \nu}(\gamma, t) - \frac{\partial u^F}{\partial \nu}(\gamma, 2T - t)\right], \quad 0 \leq t \leq T,$$

where F is so called *double* boundary control

$$F(\gamma, t) = \int\limits_0^t \mathrm{d}t' \left\{ \begin{array}{rr} f(\gamma, t'), & 0 \leq t' < T \\ -f(\gamma, 2T - t'), & T \leq t' \leq 2T \end{array} \right\}.$$

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Let a(x) be a harmonic function: $\Delta a = 0$ in Ω. Then

$$(a, u^f(\cdot, T))_{\mathcal{H}} = \int\limits_0^T \mathrm{d}t\,(T - t) \int\limits_\Gamma \mathrm{d}\Gamma \left[a(\gamma)\frac{\partial u^f}{\partial \nu}(\gamma, t) - \frac{\partial a}{\partial \nu}(\gamma)\, f(\gamma, t)\right].$$

Key elements of BCM: controllability

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Controllability (approximate): any function $a \in L_2(\Omega^T)$ may be approximated by superposition of waves $u_\alpha := u^{f_\alpha}(\cdot, T)$, $f_\alpha \in L_2(\Gamma \times [0, T])$ arbitrarily closely in L_2-metric.

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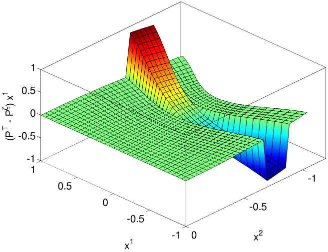
Take a smooth function $y \in \mathcal{H}$ and introduce the *projector* P^ξ on $\mathcal{H}^\xi$,

$$(P^\xi y)(x) := \left\{ \begin{array}{rr} y(x), & x \in \Omega^\xi \\ 0, & x \in \Omega \backslash \Omega^\xi \end{array} \right\}.$$

Projection $(P^T - P^\xi)\, a$ for $a = x^1$.



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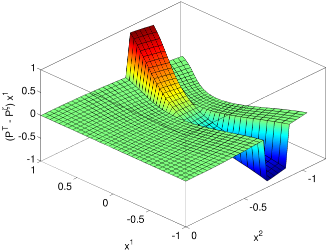

$$\forall \xi \in (0, T]: \quad P^\xi a = \sum_\alpha a_\alpha^\xi\, u_\alpha^\xi. \quad (P^\xi a, u_\beta^\xi)_{H^T} = \sum_\alpha a_\alpha^\xi\, (u_\alpha^\xi, u_\beta^\xi)_{H^T}.$$

$$A^\xi\, a^\xi = b^\xi, \quad A_{\beta\alpha}^\xi = (u_\alpha^\xi, u_\beta^\xi)_{H^T}, \quad b_\beta^\xi = (a, u_\beta^\xi)_{H^T}.$$

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We take $y_\xi^\perp$ as initial value for $(\partial v/\partial t)|_{t=T}$ in the dual system. Such discontinuous initial value produces a discontinuous wave carrying (in inverted time) a jump at its forward front along the semigeodesic rays. The jump reaches the boundary at points $x(\gamma, 0) = \gamma$ at the moment $t = T - \xi$ producing the jump of force whose amplitude can be calculated by geometric optics,

$$\left[O^T y_\xi^\perp\right](\gamma, T - \xi - 0) = \frac{\partial v^{y_\xi^\perp}}{\partial \nu}(\gamma, T - \xi - 0) = \beta(\gamma, \xi)\, y(x(\gamma, \xi)).$$

$\tilde{y}(\gamma, \xi) := \beta(\gamma, \xi)\, y(x(\gamma, \xi))$ is called an *image* of $y(x)$.

Main tool of BCM: amplitude formula

Now we combine all tricks into one tricky tool. Let $y = a(x)$,

$$O^T a_\xi^\perp = O^T (1_{\mathcal{H}} - P^\xi)\, a = O^T (P^T - P^\xi)\, a.$$

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or, using identity $O^T u^f = C^T f$,

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Reconstruction of $c(x)$

The $x^k(\gamma,\xi)$ component of the mapping can be calculated from ratio of *images* of two harmonic functions $\left\{1, x^k\right\}$:

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$$\frac{1}{\rho(\gamma,\xi)} = c^2(\gamma,\xi) = \sum_{k=1}^{n}\left[\frac{\partial x^k(\gamma,\xi)}{\partial\xi}\right]^2.$$

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The end product of BCM is the reconstructed speed of sound in cartesian coordinates $c(x) = c(x(\gamma,\xi))$.

Basis of boundary controls

The BCM assumes that boundary controls $f_\alpha(\gamma, t)$ form a basis in Sobolev space

$$\left\{ f_\alpha \in H^1(\Gamma \times [0, T]) \mid f_\alpha(\gamma, t)|_{t=0} = 0 \right\}.$$

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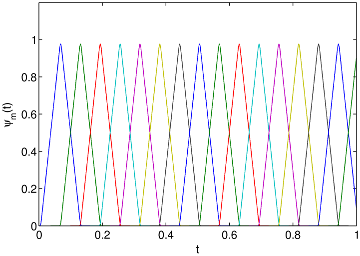

Figure: Temporal basis (budget).

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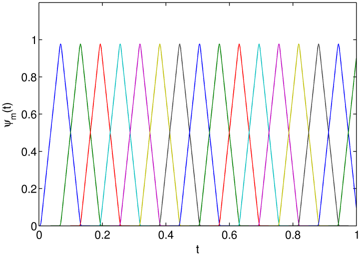

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X

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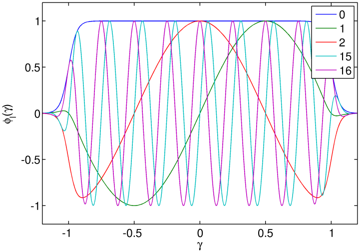

Figure: Spatial basis (trig.).

Boundary controls and system's reactions

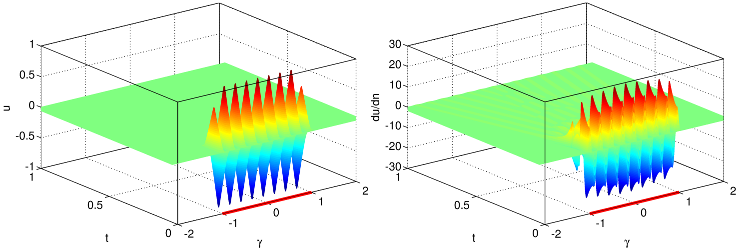

Figure: Example of basic boundary control and system's reaction.

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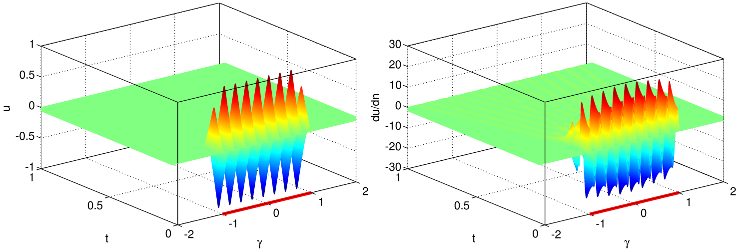

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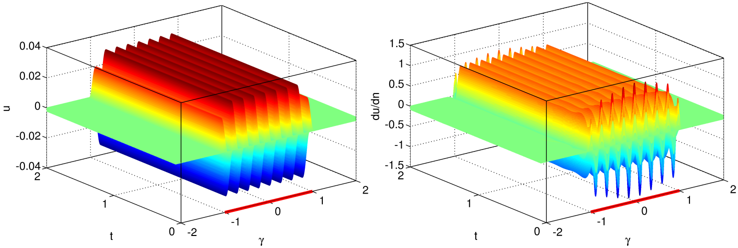

Figure: Example of double boundary control and system's reaction.

Test case 1 (basic)

We start presentation of results from the speed of sound $c(x) = \rho(x)^{-\frac{1}{2}}$ produced by Gaussian shape of density of medium,

$$\rho(x^1, x^2) = 1 + a\, g_1(x^1)\, g_2(x^2), \quad g_k(x^k) = \exp\left[-\frac{\left(x^k - \bar{x}^k\right)^2}{2\Delta_k^2}\right],$$

where $a = 1$, $\bar{x}^1 = 0$, $\bar{x}^2 = -0.5$, $\Delta_1 = 0.5$, $\Delta_2 = 0.5$.

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where $a = 1$, $\bar{x}^1 = 0$, $\bar{x}^2 = -0.5$, $\Delta_1 = 0.5$, $\Delta_2 = 0.5$.

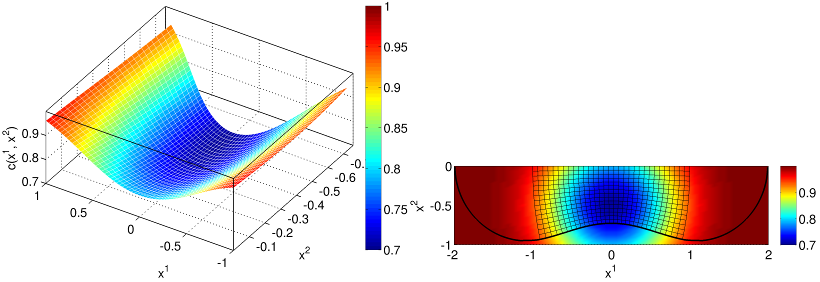


Figure: The speed of sound $c(x^1, x^2)$ (left) and semigeodesic coordinates.

Image of x^1

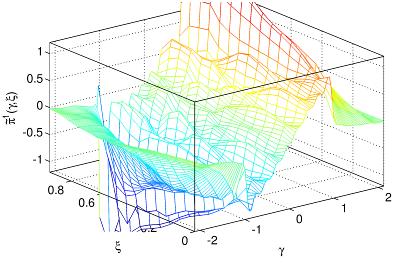
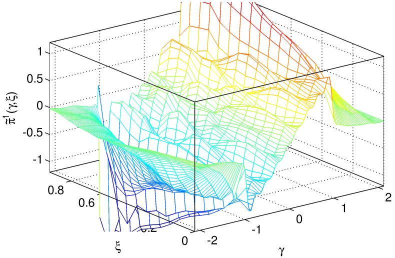


Figure: $\tilde{x}^1(\gamma, \xi)$, $\epsilon_{reg} = 10^{-5}$.

Image of x^1



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We expand discontinious projections $P^\xi a(x)$ over smooth wave solutions $u^\xi_\alpha(x, T)$ and thus observe Gibbs oscillations. The basis functions have finite resolution of order of spatio-temporal scales of the highest harmonic. All scales below the minimum ones are irrelevant, therefore we average the result of expansion over that minimum scales by convolution

$$\langle g\rangle(\gamma, t) = \int\limits_{-\infty}^{+\infty} \mathrm{d}t' \int\limits_{-\infty}^{+\infty} \mathrm{d}\gamma' K(\gamma-\gamma', t-t')\, g(\gamma', t').$$

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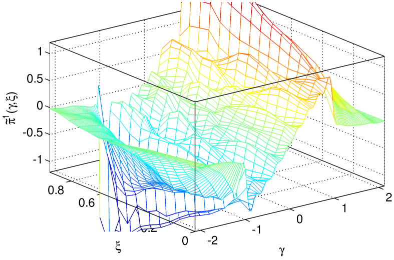

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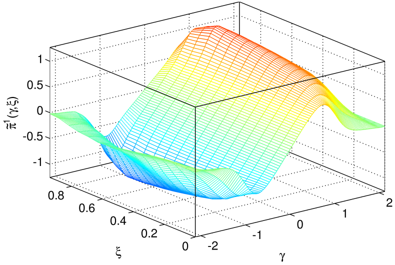

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Mapping $x = x(\gamma, \xi)$

The $x^k(\gamma, \xi)$ component of the mapping is calculated from ratio of *images* of two harmonic functions $\left\{1, x^k\right\}$:

$$\frac{\tilde{x}^k(\gamma, \xi)}{\tilde{1}(\gamma, \xi)} = \frac{\beta(\gamma, \xi) x^k(x(\gamma, \xi))}{\beta(\gamma, \xi) 1} = x^k(\gamma, \xi),$$

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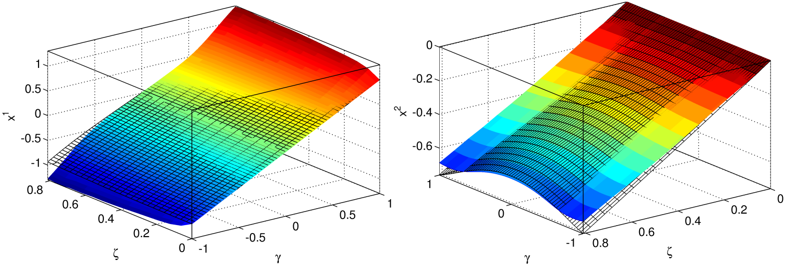

Figure: The reconstructed mapping $x = x(\gamma, \xi)$.

Speed of sound in cartesian coordinates $x = (x^1, x^2)$

The speed of sound (density) $c(\gamma, \xi)$ can be recovered from the following property of semigeodesical rays, $dl^2 = (dx^1)^2 + (dx^2)^2 = c^2\, d\xi^2$,

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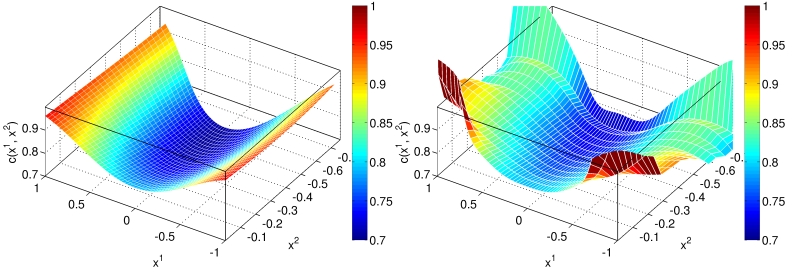

Figure: The exact (left) and reconstructed speed of sound in cartesian coordinates.

Relative error of reconstruction of speed of sound

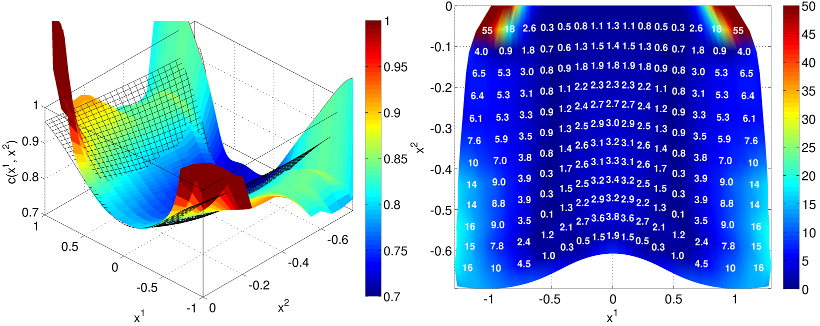

Figure: Speed of sound (left) and its relative error in percents.

Test case 2

For the second test we have selected the following density of medium,

$$\rho(x^1, x^2) = 1 - 0.5x^2 + 0.0625\left(x^1\right)^2 - a\,g_1(x^1)\,\frac{\partial g_2(x^2)}{\partial x^2},$$

where $a = 0.25$, $\bar{x}^1 = 0$, $\bar{x}^2 = -0.5$, $\Delta_1 = 0.5$, $\Delta_2 = 0.25$. The speed of sound consists of a background and two variations of order 30% of its boundary value.

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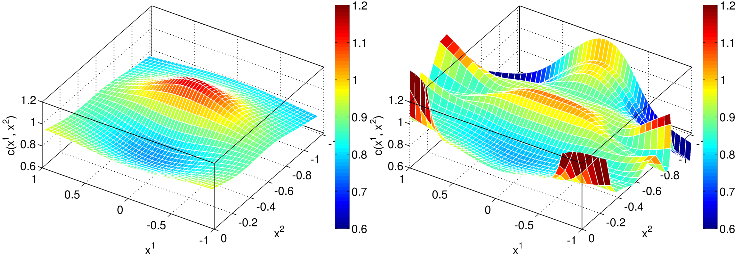
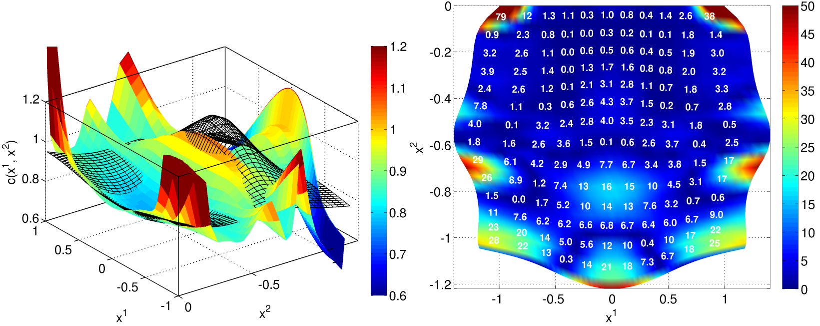


Figure: The exact (left) and recovered speed of sound $c(x^1, x^2)$.

We have used $T = 1.5$ and 16 spatial and 32 temporal functions.

Relative error of reconstruction of speed of sound



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Condition number of the matrix

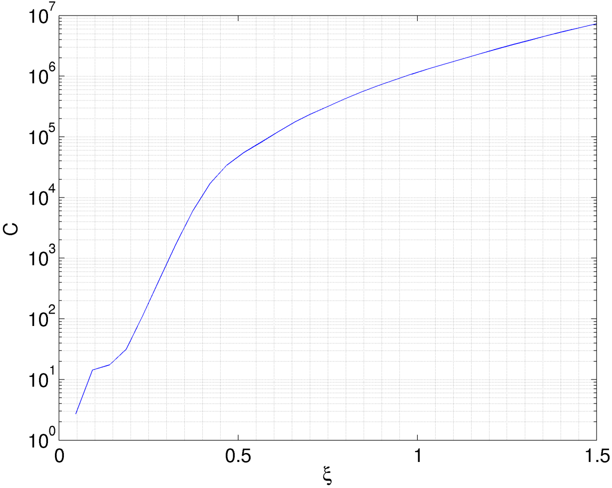

Figure: The condition number of matrix of scalar products as function of the probing time ξ: $C \sim \xi^4$.

Test case 3

Here we take

$$\rho(x^1, x^2) = 1 - 0.5x^2 + 0.0625\left(x^1\right)^2 + a\, g_1(x^1)\,\left(1 - x^2\right)\,\frac{\partial g_2(x^2)}{\partial x^2},$$

where $a = 0.25$, $\bar{x}^1 = 0$, $\bar{x}^2 = -0.5$, $\Delta_1 = 0.5$, $\Delta_2 = 0.25$. In contrast to the case 2, the corresponding speed of sound has rather strong variations (the ratio of maximum to minimum value is about 2.5).

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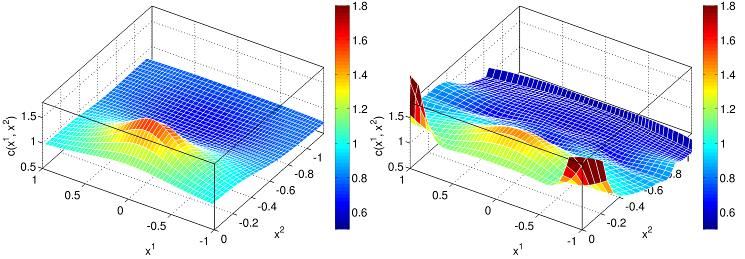

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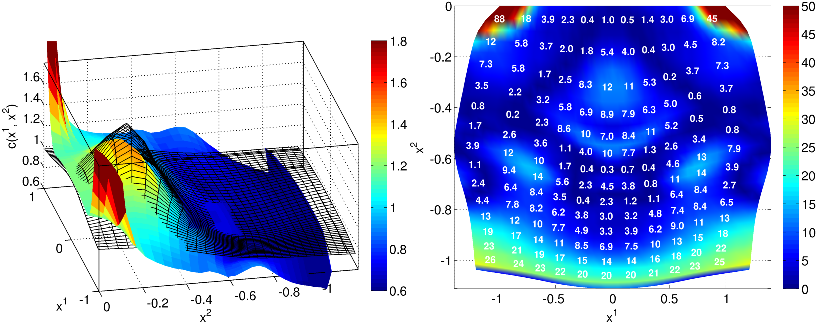

Figure: Speed of sound (left) and its relative error in percents.

Test case 4

To test the recovering algorithm for the case of sound speed quickly varying along the boundary we set

$$\rho(x^1, x^2) = 1 - a\, g_2(z^2)\, \frac{\partial g_1(z^1)}{\partial x^1},$$

$$z^1 = \cos(\phi)x^1 + \sin(\phi)(x^2 + 0.25),$$

$$z^2 = -\sin(\phi)x^1 + \cos(\phi)(x^2 + 0.25),$$

where $a = 0.25$, $\bar{x}^1 = 0$, $\bar{x}^2 = 0$, $\Delta_1 = 0.375$, $\Delta_2 = 0.25$, $\phi = \pi/12$. We have used $T = 1$ and 16 spatial and 32 temporal functions.

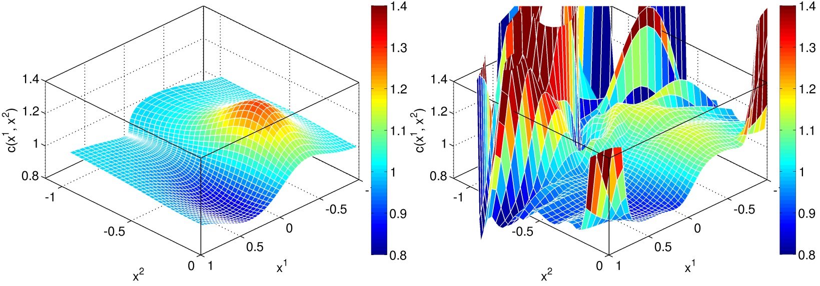

Figure: Speed of sound $c(x)$: exact values (left), recovered values

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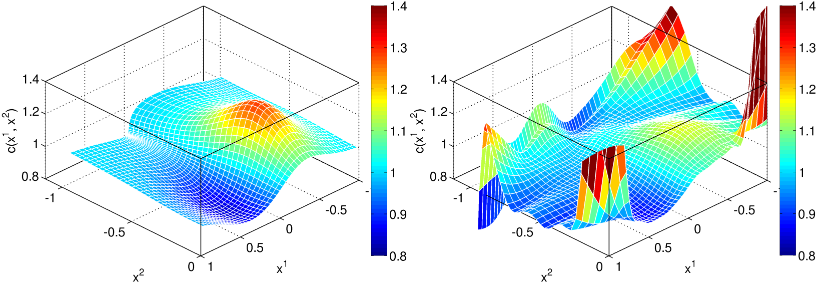

Figure: Speed of sound $c(x)$: exact values (left), pseudo-recovered values

Relative error of reconstruction of speed of sound

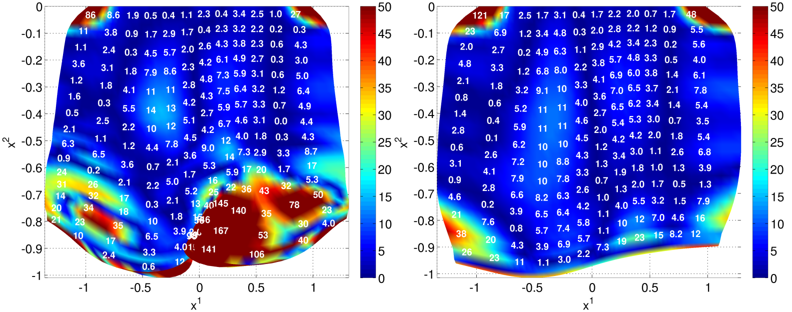

Figure: Relative errors (in percents) of the recovered sound speed $c(x)$: left - usual reconstruction, right - pseudo-reconstruction.

Test case 5

The purpose of the test is to check the ability of BCM to work with strong gradients in the recovered quantities. We prepare the density of medium as a slightly smoothed wedge with density $\rho = 5$ included in the homogeneous background with the constant density $\rho = 1$.

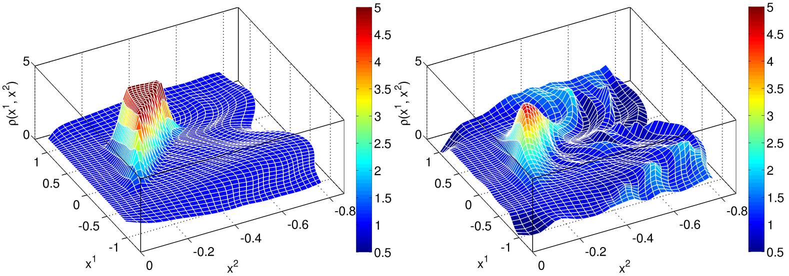

Figure: Density of medium $\rho(x)$ in the domain filled by waves initiated from σ: left - exact values, right - recovered values.

We take $T = 1$ and use the basis with 16 spatial and 32 temporal functions.

Relative error of reconstruction of speed of sound

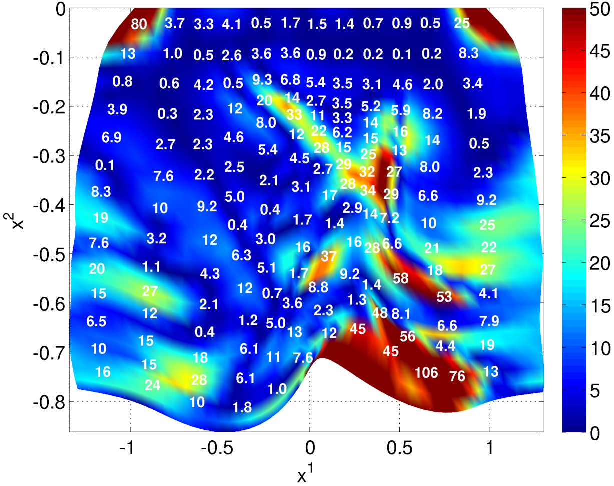

Figure: Map of relative errors of the recovered sound speed in percents.

Caustics & cut locus vs BCM

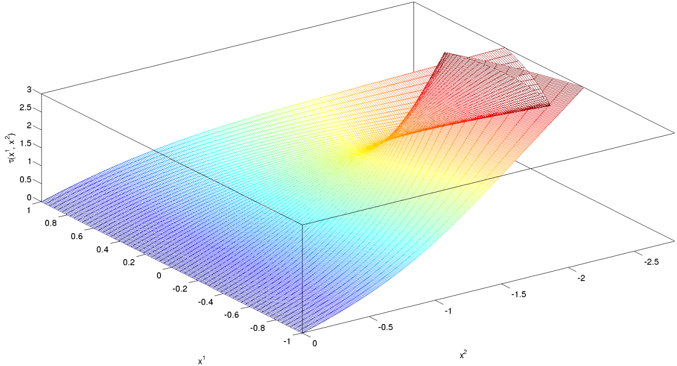

Figure: Eikonal $\tau(x^1, x^2)$ for test case 1, T $= 3$.

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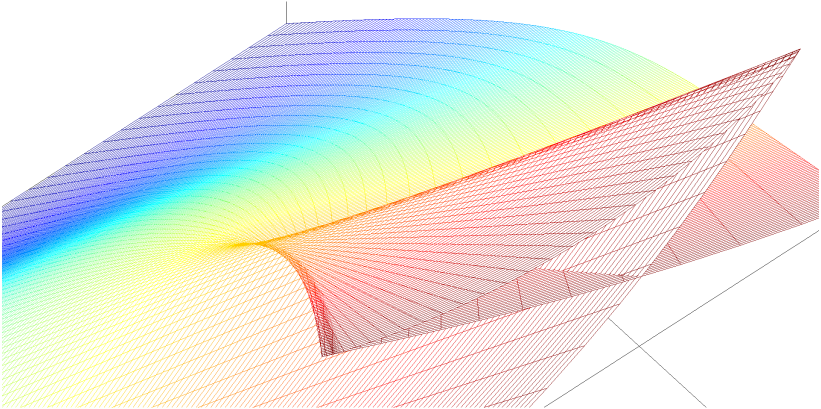

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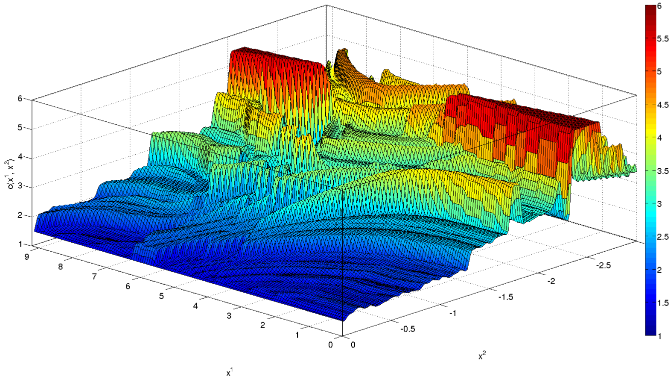
Test case 6: Marmousi model

The boundary control is done on part of boundary $x^1 \in [0, 9.2]$ km with probing time $T = 1.25$ s, basis of controls is composed from $N_t = 32$ tent-like temporal functions and $N_\gamma = 31$ tent-like spatial functions.



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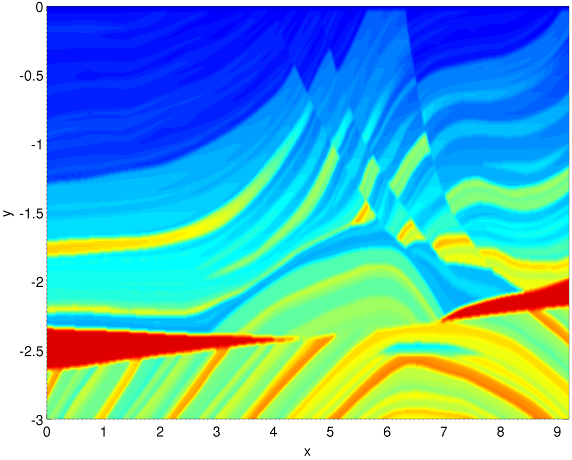

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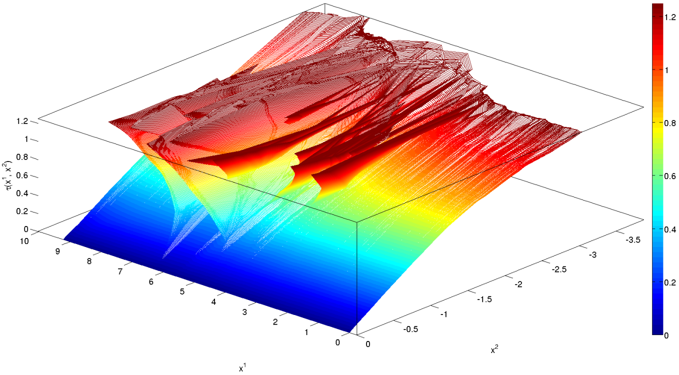

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Reconstruction of speed of sound

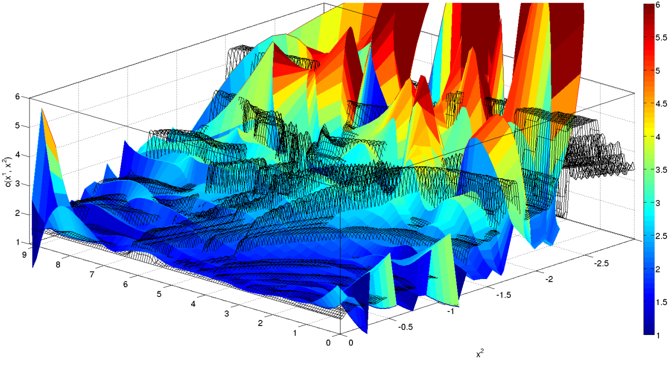

Figure: Speed of sound in Marmousi model: reconstruction ($\epsilon = 1 \cdot 10^{-5}$).

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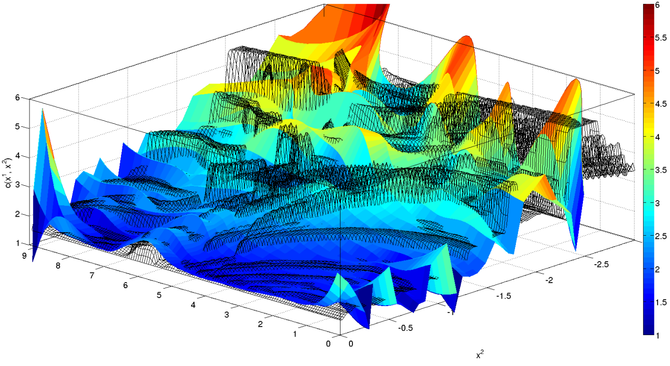


Figure: Speed of sound in Marmousi model: pseudo-reconstruction ($\epsilon = 5 \cdot 10^{-7}$).

Relative error of reconstruction of speed of sound

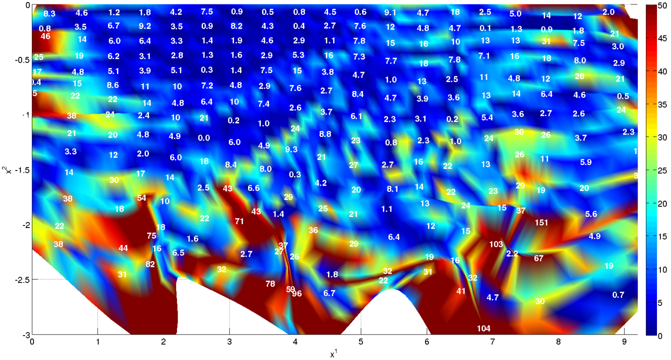
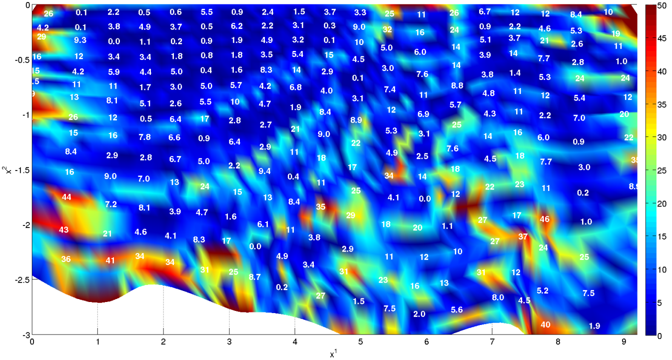


Figure: Relative error of speed of sound in percents: reconstruction ($\epsilon = 1 \cdot 10^{-5}$).

Relative error of reconstruction of speed of sound



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Conclusion

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6. The BCM is able to work with low number of spatial controls: in such a case it provides an "averaged" profile. As we hope, such a profile can be used as a starting approximation for high resolution iterative reconstruction methods.

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Thanks for your attention!